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# Low-Energy Effective Action in Non-Perturbative Electrodynamics in Curved Spacetime

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We study the heat kernel for the Laplace type partial differential operator acting on smooth sections of a complex spin-tensor bundle over a generic  $n$ -dimensional Riemannian manifold. Assuming that the curvature of the  $U(1)$  connection (that we call the electromagnetic field) is constant we compute the first two coefficients of the non-perturbative asymptotic expansion of the heat kernel which are of zero and the first order in Riemannian curvature and of arbitrary order in the electromagnetic field. We apply these results to the study of the effective action in non-perturbative electrodynamics in four dimensions and derive a generalization of the Schwinger's result for the creation of scalar and spinor particles in electromagnetic field induced by the gravitational field. We discover a new infrared divergence in the imaginary part of the effective action due to the gravitational corrections, which seems to be a new physical effect.

## 1 Introduction

The effective action is one of the most powerful tools in quantum field theory and quantum gravity (see [18, 15, 9, 10, 13]). The effective action is a functional of the background fields that encodes, in principle, all the information of quantum field theory. It determines the full one-point propagator and the full vertex functions and, hence, the whole  $S$ -matrix. Moreover, the variation of the effective action gives the effective equations for the background fields, which makes it possible to study the back-reaction of quantum processes on the classical background. In particular, the low energy effective action (or the effective potential) is the most appropriate tool for investigating the structure of the physical vacuum in quantum field theory.

The effective action is expressed in terms of the propagators and the vertex functions. One of the most powerful methods to study the propagators is the heat kernel method, which was originally proposed by Fock [16] and later generalized by Schwinger [18] who also applied it to the calculation of the one-loop effective action in quantum electrodynamics. Finally, De Witt reformulated it in the geometrical language and applied it to the case of gravitational field (see his latest book [15]).

In particular Schwinger solved exactly the case of a constant electromagnetic field and derived an heat kernel integral representation for the effective action. He showed that the heat kernel becomes a meromorphic function and a careful evaluation of the integral leads to an imaginary part of the effective action. Schwinger computed the imaginary part of the effective action and showed that it describes the effect of creation of electron-positron pairs by the electric field. This effect is now called the Schwinger mechanism. This is an essentially non-perturbative effect (non-analytic in electric field) that vanishes exponentially for weak electric fields.

Therefore its evaluation requires non-perturbative techniques for the calculation of the heat kernel in the situation when curvatures (but not their derivatives) are large (low energy approximation). A powerful approach to the calculation of the low-energy heat kernel expansion was developed in non-Abelian gauge theories and quantum gravity in [3, 4, 5, 6, 7, 14, 11, 12]. While the papers [3, 5, 6] dealt with the constant electromagnetic field in flat space, the papers [4, 7, 14] dealt with symmetric spaces (pure gravitational field in absence of an electromagnetic field). The difficulty of combining the gauge fields and gravity was finally overcome in the papers [11, 12], where homogeneous bundles with parallel curvature on symmetric spaces was studied.

In [14] we computed the heat kernel for the covariant Laplacian with a strong covariantly constant electromagnetic field in an arbitrary gravitational field. We evaluated the first three coefficients of the heat kernel asymptotic expansion in powers of Riemann curvature  $R$  but *in all orders* of the electromagnetic field  $F$ . This is equivalent to a partial summation in the heat kernel asymptotic expansion as  $t \rightarrow 0$  of all powers of  $F$  in terms which are linear and quadratic in Riemann curvature  $R$ . In the present paper we use those results to compute *explicitly* the terms linear in the Riemann curvature in the non-perturbative heat kernel expansion for the scalar and the spinor fields and compute their contribution to the imaginary part of the effective action. In other words, we generalize the Schwinger mechanism to the case of a strong electromagnetic field in a gravitational field and compute the *gravitational corrections to the original Schwinger result*.

## 2 Setup of the Problem

Let  $M$  be a  $n$ -dimensional compact Riemannian manifold (with positive-definite metric  $g_{\mu\nu}$ ) without boundary and  $S$  be a complex spin-tensor vector bundle over  $M$  realizing a representation of the group  $\text{Spin}(n) \otimes U(1)$ . Let  $\varphi$  be a section of the bundle  $S$  and  $\nabla$  be the total connection on the bundle  $S$  (including the spin connection as well as the  $U(1)$ -connection). Then the commutator of covariant derivatives defines the curvatures

$$[\nabla_\mu, \nabla_\nu]\varphi = (\mathcal{R}_{\mu\nu} + iF_{\mu\nu})\varphi, \quad (2.1)$$

where  $F_{\mu\nu}$  is the curvature of the  $U(1)$ -connection (which will be also called the electromagnetic field) and  $\mathcal{R}_{\mu\nu}$  is the curvature of the spin connection defined by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}R_{\mu\nu}^{ab}\Sigma_{ab}, \quad (2.2)$$

with  $\Sigma_{ab}$  being the generators of the spin group  $\text{Spin}(n)$  satisfying the commutation relations

$$[\Sigma_{ab}, \Sigma^{cd}] = 4\delta_{[a}^{[c}\Sigma^{d]}_{b]}. \quad (2.3)$$

Note that for the scalar fields  $\mathcal{R}_{\mu\nu} = 0$  and for the spinor fields

$$\Sigma_{ab} = \frac{1}{2}\gamma_{ab}, \quad (2.4)$$

where  $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$  (more generally, we define  $\gamma_{a_1\dots a_m} = \gamma_{[a_1}\dots\gamma_{a_m]}$ ) and  $\gamma_a$  are the Dirac matrices generating the Clifford algebra

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2g_{ab}\mathbb{I}. \quad (2.5)$$

## 2.1 Differential Operators

In the present paper we consider a second-order Laplace type partial differential operator,

$$L = -\Delta + \xi R + Q, \quad (2.6)$$

where  $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$  is the Laplacian,  $\xi$  is a constant parameter, and  $Q$  is a smooth endomorphism of the bundle  $\mathcal{S}$ . This operator is elliptic and self-adjoint and has a positive-definite leading symbol. Usually, for scalar fields we set

$$Q^{\text{scalar}} = 0. \quad (2.7)$$

Moreover, for canonical scalar fields the coupling

$$\xi^{\text{scalar}} = \begin{cases} 0 & \text{for canonical scalar fields,} \\ \frac{(n-2)}{4(n-1)} & \text{for conformal scalar fields.} \end{cases} \quad (2.8)$$

Another important case is the square of the Dirac operator acting on spinor fields

$$L = D^2, \quad (2.9)$$

where

$$D = i\gamma^\mu \nabla_\mu. \quad (2.10)$$

It is easy to see that in this case we have

$$\xi^{\text{spinor}} = \frac{1}{4} \quad (2.11)$$

and

$$Q^{\text{spinor}} = -\frac{1}{2} i F_{\mu\nu} \gamma^{\mu\nu}. \quad (2.12)$$

## 2.2 Effective Action

The object of primary interest in quantum field theory is the (Euclidean) one-loop effective action determined by the formal determinant

$$\Gamma_{(1)} = \sigma \log \text{Det} (L + m^2), \quad (2.13)$$

where  $\sigma$  is the fermion number of the field equal to  $(+1)$  for boson fields and  $(-1)$  for fermion fields,  $m$  is a mass parameter, which is assumed to be sufficiently large

so that the operator  $(L + m^2)$  is positive. Notice that the usual factor  $\frac{1}{2}$  is missing because the field is complex, which is equivalent to the contribution of two real fields. Of course, this formal expression is divergent. To rigorously define the determinant of a differential operator one needs to introduce some regularization and then to renormalize it. One of the best ways to do it is via the heat kernel method.

In a physical theory the effective action describes the in-out vacuum transition amplitude via

$$\langle \text{out} | \text{in} \rangle = \exp[i\Gamma_{(1)}] . \quad (2.14)$$

The real part of the effective action describes the polarization of vacuum of quantum fields by the background fields and the imaginary part describes the creation of particles. Namely, the probability of particles production (in the whole space-time) is given by

$$P = 1 - |\langle \text{out} | \text{in} \rangle|^2 = 1 - \exp[-2 \text{Im} \Gamma_{(1)}] . \quad (2.15)$$

Of course, the unitarity requires that the imaginary part of the effective action should be positive

$$\text{Im} \Gamma_{(1)} \geq 0 . \quad (2.16)$$

Notice that usually, when the imaginary part of the effective action is small, we just have

$$P \approx 2 \text{Im} \Gamma_{(1)} . \quad (2.17)$$

The one-loop effective Lagrangian is defined by

$$\Gamma_{(1)} = \int_M dx g^{1/2} \mathcal{L} . \quad (2.18)$$

Therefore, the rate of the particles production per unit volume per unit time is given by the imaginary part of the effective Lagrangian

$$R = \frac{P}{VT} \approx 2 \text{Im} \mathcal{L} . \quad (2.19)$$

## 2.3 Spectral Functions

The heat kernel for the operator  $L$  is defined as the solution of the heat equation

$$(\partial_t + L) U(t|x, x') = 0 , \quad (2.20)$$

with the initial condition

$$U(0|x, x') = \delta(x, x') . \quad (2.21)$$

where  $\delta(x, x')$  is the covariant scalar delta function on the bundle  $\mathcal{S}$ . The heat kernel diagonal is defined by

$$U^{\text{diag}}(t) = U(t|x, x) . \quad (2.22)$$

One of the best ways to describe the spectral properties of the operator  $L$  is via the heat trace

$$\text{Tr} \exp(-tL) = \int_M dx g^{1/2} \Theta(t) , \quad (2.23)$$

where  $dx$  is the Lebesgue measure on the manifold  $M$ ,  $g = \det g_{\mu\nu}$  and

$$\Theta(t) = \text{tr} U^{\text{diag}}(t) . \quad (2.24)$$

Here  $\text{tr}$  denotes the fiber trace over the bundle  $\mathcal{S}$ .

The determinant of the operator can be defined within the so-called zeta-function regularization as follows. First, one defines the zeta function by

$$\zeta(s) = \mu^{2s} \text{Tr} (L + m^2)^{-s} = \int_M dx g^{1/2} Z(s) , \quad (2.25)$$

where

$$Z(s) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-tm^2} \Theta(t) , \quad (2.26)$$

and  $\mu$  is a renormalization parameter introduced to preserve dimensions. The zeta function  $\zeta(s)$  is a meromorphic function of  $s$  analytic at  $s = 0$ . This enables one to define the (zeta-regularized) functional determinant of the operator  $(L + m^2)$  by

$$\text{Det} (L + m^2) = \exp[-\zeta'(0)] , \quad (2.27)$$

where  $\zeta'(s) = \frac{d}{ds} \zeta(s)$ . Therefore, the one-loop effective action is simply

$$\Gamma_{(1)} = -\sigma \zeta'(0) , \quad (2.28)$$

and the one-loop effective Lagrangian is given by

$$\mathcal{L} = -\sigma Z'(0) . \quad (2.29)$$

The effective Lagrangian can be also defined simply in the cut-off regularization by

$$\mathcal{L} = -\sigma \int_{\varepsilon\mu^2}^{\infty} \frac{dt}{t} e^{-tm^2} \Theta(t), \quad (2.30)$$

where  $\varepsilon$  is a regularization parameter, which should be set to zero after subtracting the divergent terms. Another regularization is the dimensional regularization, in which one simply defines the effective action by the formal integral

$$\mathcal{L} = -\sigma \mu^{2\varepsilon} \int_0^{\infty} \frac{dt}{t} e^{-tm^2} \Theta(t), \quad (2.31)$$

where the heat trace is formally computed in complex dimension  $(n - 2\varepsilon)$  with sufficiently large real part of  $\varepsilon$  so that the integral is finite. The renormalized effective action is obtained then by subtracting the simple pole in  $\varepsilon$ .

For elliptic operators (in the Euclidean setup) the heat trace is a smooth function of  $t$ ; in many cases it is even an analytic function of  $t$  in the neighborhood of the positive real axis. However, in the physical case for hyperbolic operators (in the Lorentzian setup) the heat trace can have singularities even on the positive real axis of  $t$ . As we will show later in the approximation under consideration (for constant electromagnetic field) it becomes a meromorphic function of  $t$  with an essential singularity at  $t = 0$  and some poles  $t_k$ ,  $k = 1, 2, \dots$ , on the positive real axis. It turns out that the imaginary part of the effective action does not depend on the regularization method and is uniquely defined by the contribution of these poles. These poles should be avoided from above, which gives

$$\text{Im } \mathcal{L} = -\sigma\pi \sum_{k=1}^{\infty} \text{Res} \left\{ t^{-1} e^{-tm^2} \Theta(t); t_k \right\}. \quad (2.32)$$

This method was first elaborated and used by Schwinger [18] in quantum electrodynamics to calculate the electron-positron pair production by a constant electric field. One of the goal of our work is to *generalize the Schwinger results* for the case of constant electromagnetic field in a gravitational field. We will compute the *extra contribution* to the particle production by a constant electromagnetic field *induced by the gravitational field*.

## 2.4 Heat Kernel Asymptotic Expansion

It is well known [17] that the heat kernel diagonal has the asymptotic expansion as  $t \rightarrow 0$  (see also [2, 8, 9, 19])

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k a_k , \quad (2.33)$$

where  $a_k$  are the local heat kernel coefficients. Then the trace of the diagonal heat kernel has the corresponding asymptotic expansion

$$\Theta(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k A_k , \quad (2.34)$$

where

$$A_k = \text{tr } a_k . \quad (2.35)$$

The diagonal heat kernel coefficients  $a_k$  are polynomials in the jets of the metric, the  $U(1)$ -connection and the potential term  $Q$ , that is, in the curvature tensors,  $Q$ , and their derivatives. The lower order diagonal heat kernel coefficients are well known [17, 2, 9]

$$a_0 = 1 , \quad (2.36)$$

$$a_1 = -Q + \left( \frac{1}{6} - \xi \right) R . \quad (2.37)$$

To avoid confusion we should stress that the normalization of the coefficients  $a_k$  differs from the papers [2, 8, 9].

In our previous paper [14] we studied the case of a *parallel*  $U(1)$  curvature (covariantly constant electromagnetic field), i.e.

$$\nabla_\mu F_{\alpha\beta} = 0 . \quad (2.38)$$

In the present paper we will also assume that the potential term  $Q$  is covariantly constant too

$$\nabla_\mu Q = 0 . \quad (2.39)$$

By summing up all powers of  $F$  in the asymptotic expansion of the heat kernel diagonal we obtained a *new (non-perturbative) asymptotic expansion*

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} \exp(-tQ) J(t) \sum_{k=0}^{\infty} t^k b_k(t) , \quad (2.40)$$



where

$$J(t) = \det \left( \frac{tiF}{\sinh(tiF)} \right)^{1/2} \quad (2.41)$$

and  $b_k(t)$  are the modified heat kernel coefficients which are analytic functions of  $t$  at  $t = 0$  which depend on  $F$  only in the dimensionless combination  $tF$ . Here and everywhere below all functions of the 2-form  $F$  are analytic at 0 and should be understood in terms of a power series in the matrix  $F = (F^\mu{}_\nu)$ . Notice the *position of indices* here, it is important! There is a difference here between Euclidean case and the Lorentzian one since the raising of indices by a Minkowski metric does change the properties of the matrix  $F$ . Also, here  $\det$  denotes the determinant with respect to the tangent space indices.

The fiber trace of the heat kernel diagonal has then the asymptotic expansion

$$\Theta(t) \sim (4\pi t)^{-n/2} \Phi(t) \sum_{k=0}^{\infty} t^k B_k(t), \quad (2.42)$$

where

$$\Phi(t) = J(t) \text{tr} \exp(-tQ), \quad (2.43)$$

$$B_k(t) = \frac{\text{tr} \exp(-tQ) b_k(t)}{\text{tr} \exp(-tQ)} \quad (2.44)$$

are new (*non-perturbative*) heat kernel coefficients of the operator  $L$ . The integrals  $\int_M dx g^{1/2} B_k(t)$  are then the spectral invariants of the operator  $L$ .

This expansion can be described more rigorously as follows. We rescale the  $U(1)$ -curvature  $F$  by

$$F \mapsto F(t) = t^{-1} \tilde{F}, \quad (2.45)$$

so that  $tF(t) = \tilde{F}$  is independent of  $t$ . Then the operator  $L(t)$  becomes dependent on  $t$  (in a singular way!). However, the heat trace still has a nice asymptotic expansion (2.42) as  $t \rightarrow 0$ , where the coefficients  $B_k$  are expressed in terms of  $\tilde{F} = tF(t)$ , and, therefore, are independent of  $t$ . Thus, what we are doing is the *asymptotic expansion of the heat trace for a particular case of a singular (as  $t \rightarrow 0$ ) time-dependent operator  $L(t)$* .

### 3 Calculation of the Coefficient $B_1(t)$

For the first two coefficients we obtained [14]

$$b_0(t) = 1, \quad (3.1)$$

$$b_1(t) = \left\{ \Sigma_{\mu\alpha} W_{\nu\beta}(t) + V_{\mu\alpha\nu\beta}(t) \right\} R^{\mu\alpha\nu\beta} \quad (3.2)$$

where

$$W(t) = \frac{1}{2} \left( \coth(tiF) - \frac{1}{tiF} \right), \quad (3.3)$$

$$\begin{aligned} V^{\mu\alpha}_{\nu\beta}(t) = & \left( \frac{1}{3} - \xi \right) \delta^{[\mu}_{\nu} \delta^{\alpha]}_{\beta} + \int_0^1 d\tau \left\{ -\frac{1}{24} \mathcal{B}^{[\mu}_{[\nu}(\tau) \mathcal{Z}^{\alpha]}_{\beta]}(\tau) + \frac{1}{6} \mathcal{A}^{[\mu\alpha]}(\tau) \mathcal{A}_{[\nu\beta]}(\tau) \right. \\ & \left. - \frac{1}{12} \mathcal{A}^{[\mu}_{[\nu}(\tau) \mathcal{A}^{\alpha]}_{\beta]}(\tau) - \frac{1}{4} \mathcal{A}^{[\mu}_{[\nu}(\tau) \mathcal{A}_{\beta]}^{\alpha]}(\tau) \right\}, \end{aligned} \quad (3.4)$$

and

$$\mathcal{A}(\tau) = \frac{1}{2} \frac{\exp[(1-2\tau)tiF] - \exp(-tiF)}{\sinh(tiF)}, \quad (3.5)$$

$$\mathcal{B}(\tau) = \frac{\coth(tiF)}{tiF} - \frac{1}{tiF \sinh(tiF)} \cosh[(1-2\tau)tiF], \quad (3.6)$$

$$\mathcal{Z}(\tau) = 3tiF \coth(tiF) + \frac{tiF}{\sinh(tiF)} \cosh[(1-2\tau)tiF]. \quad (3.7)$$

The trace coefficients are then given by

$$B_0(t) = 1, \quad (3.8)$$

$$B_1(t) = \left\{ \Psi_{\mu\alpha}(t) W_{\nu\beta}(t) + V_{\mu\alpha\nu\beta}(t) \right\} R^{\mu\alpha\nu\beta}, \quad (3.9)$$

where

$$\Psi(t)_{\mu\alpha} = \frac{\text{tr} \exp(-tQ) \Sigma_{\mu\alpha}}{\text{tr} \exp(-tQ)}. \quad (3.10)$$

### 3.1 Spectral Decomposition

To evaluate it we use the spectral decomposition of the matrix  $F = (F^\mu_{\nu})$ ,

$$F = \sum_{k=1}^N B_k E_k, \quad (3.11)$$

where  $B_k$  are some real invariants and  $E_k = (E_k^\mu_{\nu})$  are some matrices satisfying the equations

$$E_{k\mu\nu} = -E_{k\nu\mu}, \quad (3.12)$$

$$E_{\mu[\nu}^k E_{\alpha\beta]}^k = 0, \quad (3.13)$$

and for  $k \neq m$

$$E_k E_m = 0. \quad (3.14)$$

Here, of course,  $N \leq [n/2]$ . The invariants  $B_k$  (that we call “magnetic fields”) should not be confused with the heat trace coefficients  $B_0$  and  $B_1$ .

Next, we define the matrices  $\Pi_k = (\Pi_k^\mu{}_\nu)$  by

$$\Pi_k = -E_k^2. \quad (3.15)$$

They satisfy the equations

$$\Pi_{k\mu\nu} = \Pi_{k\nu\mu}, \quad (3.16)$$

$$E_k \Pi_k = \Pi_k E_k = E_k, \quad (3.17)$$

and for  $k \neq m$

$$E_k \Pi_m = \Pi_m E_k = 0, \quad \Pi_k \Pi_m = 0. \quad (3.18)$$

To compute functions of the matrix  $F$  we need to know its eigenvalues. We distinguish two different cases.

**Euclidean Case.** In this case the metric has Euclidean signature  $(+ + \cdots +)$  and the non-zero eigenvalues of the matrix  $F$  are  $\pm iB_1, \dots, \pm iB_N$ , (which are all imaginary). Of course, it may also have a number of zero eigenvalues. In this case the matrices  $\Pi_k$  are nothing but the projections on 2-dimensional eigenspaces satisfying

$$\Pi_k^2 = \Pi_k, \quad \Pi_k^\mu{}_\mu = 2. \quad (3.19)$$

In this case we also have

$$B_k = \frac{1}{2} E_k^{\mu\nu} F_{\mu\nu}. \quad (3.20)$$

Then we have

$$(iF)^{2m} = \sum_{k=1}^N B_k^{2m} \Pi_k, \quad (m \geq 1) \quad (3.21)$$

$$(iF)^{2m+1} = \sum_{k=1}^N B_k^{2m+1} iE_k, \quad (m \geq 0), \quad (3.22)$$

and, therefore, for any analytic function of  $tiF$  at  $t = 0$  we have

$$f(tiF) = f(0)\mathbb{I} + \sum_{k=1}^N \left\{ \frac{1}{2} \left[ f(tB_k) + f(-tB_k) - 2f(0) \right] \Pi_k + \frac{1}{2} \left[ f(tB_k) - f(-tB_k) \right] iE_k \right\}. \quad (3.23)$$

**Pseudo-Euclidean Case.** This is the physically relevant case of pseudo-Euclidean (Lorentzian) metric with the signature  $(- + \cdots +)$ . Then the non-zero eigenvalues of the matrix  $F$  are  $\pm B_1$  (which are real) and  $\pm iB_2, \dots, \pm iB_N$ , (which are imaginary). We will call the invariant  $B_1$ , determining the real eigenvalue, the “electric field” and denote it by  $B_1 = E$ , and the invariants  $B_k, k = 2, \dots, N$ , determining the imaginary eigenvalues, the “magnetic fields”. So, in general, there is one electric field and  $(N - 1)$  magnetic fields. Again, there may be some zero eigenvalues as well.

In this case the matrices  $\Pi_2, \dots, \Pi_N$  are the orthogonal eigen-projections as before, but the matrix  $\Pi_1$  is equal to the negative of the corresponding projection, in particular,

$$\Pi_1^2 = -\Pi_1, \quad \Pi_1 E_1 = -E_1, \quad \Pi_1^\mu{}_\mu = -2. \quad (3.24)$$

Now, we have

$$(iF)^{2m} = -(iE)^{2m}\Pi_1 + \sum_{k=2}^N B_k^{2m}\Pi_k, \quad (m \geq 1) \quad (3.25)$$

$$(iF)^{2m+1} = (iE)^{2m+1}E_1 + \sum_{k=2}^N B_k^{2m+1}iE_k, \quad (m \geq 0), \quad (3.26)$$

Thus, to obtain the results for the pseudo-Euclidean case from the result for the Euclidean case we should just substitute formally

$$B_1 \mapsto iE, \quad iE_1 \mapsto E_1, \quad \Pi_1 \mapsto -\Pi_1. \quad (3.27)$$

In this way, we obtain for an analytic function of  $itF$ ,

$$\begin{aligned} f(itF) &= f(0)\mathbb{I} - \frac{1}{2}\left[f(itE) + f(-itE) - 2f(0)\right]\Pi_1 + \frac{1}{2}\left[f(itE) - f(-itE)\right]E_1 \\ &\quad + \sum_{k=2}^N \left\{ \frac{1}{2}\left[f(itB_k) + f(-itB_k) - 2f(0)\right]\Pi_k + \frac{1}{2}\left[f(itB_k) - f(-itB_k)\right]iE_k \right\}. \end{aligned} \quad (3.28)$$

### 3.2 Scalar and Spinor Fields

First of all, we note that for scalar fields

$$\Phi^{\text{scalar}}(t) = J(t), \quad \Psi_{\mu\nu}^{\text{scalar}}(t) = 0. \quad (3.29)$$

For the spinor fields we have

$$\Phi^{\text{spinor}}(t) = J(t) \text{tr} \exp\left(\frac{1}{2}tiF_{\mu\nu}\gamma^{\mu\nu}\right) \quad (3.30)$$

$$\Psi_{\alpha\beta}^{\text{spinor}}(t) = \frac{1}{2} \frac{\text{tr} \gamma_{\alpha\beta} \exp\left(\frac{1}{2}tiF_{\mu\nu}\gamma^{\mu\nu}\right)}{\text{tr} \exp\left(\frac{1}{2}tiF_{\rho\sigma}\gamma^{\rho\sigma}\right)}. \quad (3.31)$$

Here  $\text{tr}$  denotes the trace with respect to the spinor indices.

We will compute these functions as follows. We define the matrices

$$T_k = \frac{1}{2}iE_k^{\mu\nu}\gamma_{\mu\nu}. \quad (3.32)$$

Then by using the properties of the matrices  $E_k$  and the product of the matrices  $\gamma_{\mu\nu}$

$$\gamma^{\mu\nu}\gamma_{\alpha\beta} = \gamma^{\mu\nu}{}_{\alpha\beta} - 4\delta_{[\alpha}^{[\mu}\gamma^{\nu]}_{\beta]} - 2\delta_{\alpha}^{[\mu}\delta_{\beta}^{\nu]}\mathbb{I}, \quad (3.33)$$

(and some other properties of Dirac matrices in  $n$  dimensions) one can show that these matrices are mutually commuting involutions, that is,

$$T_k^2 = \mathbb{I}, \quad (3.34)$$

and

$$[T_k, T_m] = 0. \quad (3.35)$$

Also, the product of two different matrices is (for  $k \neq m$ )

$$T_k T_m = -\frac{1}{4}E_k^{\mu\nu}E_m^{\alpha\beta}\gamma_{\mu\nu\alpha\beta}. \quad (3.36)$$

More generally, the product of  $m > 1$  different matrices is

$$T_{k_1} \cdots T_{k_m} = \left(\frac{i}{2}\right)^m E_{k_1}^{\mu_1\mu_2} \cdots E_{k_m}^{\mu_{2m-1}\mu_{2m}} \gamma_{\mu_1 \dots \mu_{2m}}. \quad (3.37)$$

It is well known that the matrices  $\gamma_{\mu_1 \dots \mu_k}$  are traceless for any  $k$  and the trace of the product of two matrices  $\gamma_{\mu_1 \dots \mu_k}$  and  $\gamma_{\nu_1 \dots \nu_m}$  is non-zero only for  $k = m$ . By using these properties we obtain the traces

$$\text{tr} T_k = 0, \quad (3.38)$$

$$\text{tr} \gamma^{\alpha\beta} T_k = -2^{[n/2]} i E_k^{\alpha\beta}, \quad (3.39)$$

and for  $m > 1$ :

$$\text{tr } T_{k_1} \cdots T_{k_m} = 0, \quad (3.40)$$

$$\text{tr } \gamma^{\alpha\beta} T_{k_1} \cdots T_{k_m} = 0. \quad (3.41)$$

when all indices  $k_1, \dots, k_m$  are different.

Now, by using the spectral decomposition of the matrix  $F$  we easily obtain first

$$J(t) = \prod_{k=1}^N \frac{tB_k}{\sinh(tB_k)}. \quad (3.42)$$

and

$$\text{tr } \exp\left(\frac{1}{2}tiF_{\mu\nu}\gamma^{\mu\nu}\right) = \text{tr } \prod_{k=1}^N \exp(tT_k B_k). \quad (3.43)$$

$$\text{tr } \gamma^{\alpha\beta} \exp\left(\frac{1}{2}tiF_{\mu\nu}\gamma^{\mu\nu}\right) = \text{tr } \gamma^{\alpha\beta} \prod_{k=1}^N \exp(tT_k B_k). \quad (3.44)$$

By using the properties of the matrices  $T_k$  we get

$$\exp(tT_k B_k) = \cosh(tB_k) + T_k \sinh(tB_k). \quad (3.45)$$

Therefore

$$\text{tr } \exp\left(\frac{1}{2}tiF_{\mu\nu}\gamma^{\mu\nu}\right) = 2^{[n/2]} \prod_{k=1}^N \cosh(tB_k), \quad (3.46)$$

and

$$\begin{aligned} \text{tr } \gamma^{\alpha\beta} \prod_{k=1}^N \exp(tT_k B_k) &= \prod_{j=1}^N \cosh(tB_j) \sum_{k=1}^N \tanh(tB_k) \text{tr } \gamma^{\alpha\beta} T_k \\ &= -2^{[n/2]} \prod_{j=1}^N \cosh(tB_j) \sum_{k=1}^N \tanh(tB_k) iE_k^{\alpha\beta}. \end{aligned} \quad (3.47)$$

Thus for the spinor fields

$$\Phi^{\text{spinor}}(t) = 2^{[n/2]} \prod_{k=1}^N tB_k \coth(tB_k), \quad (3.48)$$

and

$$\Psi_{\alpha\beta}^{\text{spinor}}(t) = -\frac{1}{2} \sum_{k=1}^N \tanh(tB_k) iE_{k\alpha\beta}. \quad (3.49)$$

By the way, this simply means that

$$\Psi^{\text{spinor}}(t) = -\frac{1}{2} \tanh(tiF). \quad (3.50)$$

### 3.3 Calculation of the Tensor $V_{\mu\alpha\nu\beta}(t)$

Next, we compute the tensor  $V^{\mu\alpha}_{\nu\beta}(t)$ . First, we rewrite in the form

$$\begin{aligned} V^{\mu\alpha}_{\nu\beta}(t) = & \left( \frac{1}{3} - \xi \right) \delta^{[\mu}_{\nu} \delta^{\alpha]}_{\beta} + \int_0^1 d\tau \left\{ -\frac{1}{24} \mathcal{B}^{[\mu}_{\nu}(\tau) \mathcal{Z}^{\alpha]}_{\beta]}(\tau) \right. \\ & \left. + \frac{1}{16} \mathcal{X}^{\mu\alpha}(\tau) \mathcal{X}_{\nu\beta}(\tau) - \frac{1}{12} \mathcal{Y}^{[\mu}_{\nu}(\tau) \mathcal{Y}^{\alpha]}_{\beta]}(\tau) \right\}, \end{aligned} \quad (3.51)$$

where

$$\mathcal{X}(\tau) = -\coth(tiF) + \frac{\cosh[(1-2\tau)tiF]}{\sinh(tiF)}, \quad (3.52)$$

$$\mathcal{Y}(\tau) = \mathbb{I} + \frac{\sinh[(1-2\tau)tB_k]}{\sinh(tB_k)}. \quad (3.53)$$

Next, we parametrize these matrices as follows

$$\mathcal{B}(\tau) = 2\tau(1-\tau)\mathbb{I} + \sum_{k=1}^N f_{1,k}(\tau)\Pi_k, \quad (3.54)$$

$$\mathcal{Z}(\tau) = 4\mathbb{I} + \sum_{k=1}^N f_{2,k}(\tau)\Pi_k, \quad (3.55)$$

$$\mathcal{Y}(\tau) = 2(1-\tau)\mathbb{I} + \sum_{k=1}^N f_{3,k}(\tau)\Pi_k, \quad (3.56)$$

$$\mathcal{X}(\tau) = \sum_{k=1}^N f_{4,k}(\tau)iE_k, \quad (3.57)$$

$$W(t) = \sum_{k=1}^N f_{5,k}(t)iE_k, \quad (3.58)$$

where

$$f_{1,k}(\tau) = \frac{\coth(tB_k)}{tB_k} - \frac{1}{tB_k \sinh(tB_k)} \cosh[(1-2\tau)tB_k] - 2\tau(1-\tau), \quad (3.59)$$

$$f_{2,k}(\tau) = 3tB_k \coth(tB_k) + \frac{tB_k}{\sinh(tB_k)} \cosh[(1-2\tau)tB_k] - 4, \quad (3.60)$$

$$f_{3,k}(\tau) = \frac{\sinh[(1-2\tau)tB_k]}{\sinh(tB_k)} - (1-2\tau), \quad (3.61)$$

$$f_{4,k}(\tau) = -\coth(tB_k) + \frac{\cosh[(1-2\tau)tB_k]}{\sinh(tB_k)}, \quad (3.62)$$

$$f_{5,k}(t) = \frac{1}{2} \left( \coth(tB_k) - \frac{1}{tB_k} \right). \quad (3.63)$$

This parametrization is convenient because all functions  $f_{m,k}(\tau)$  are analytic functions of  $t$  at  $t = 0$  and  $f_{m,k}(\tau)|_{t=0} = 0$ .

Then we obtain

$$\begin{aligned} V^{\mu\alpha}_{\nu\beta}(t) = & \left( \frac{1}{6} - \xi \right) \delta^{[\mu}_{[\nu} \delta^{\alpha]}_{\beta]} + \sum_{k=1}^N \varphi_k(t) \Pi_k^{[\mu}_{[\nu} \delta^{\alpha]}_{\beta]} \\ & + \sum_{k=1}^N \sum_{m=1}^N \left[ \rho_{km}(t) \Pi_k^{[\mu}_{[\nu} \Pi_m^{\alpha]}_{\beta]} - \sigma_{km}(t) E_k^{\mu\alpha} E_m{}_{\nu\beta} \right], \end{aligned} \quad (3.64)$$

where

$$\varphi_k(t) = -\frac{1}{12} \int_0^1 d\tau [2f_{1,k}(\tau) + \tau(1-\tau)f_{2,k}(\tau) + 4(1-\tau)f_{3,k}(\tau)], \quad (3.65)$$

$$\rho_{km}(t) = -\frac{1}{48} \int_0^1 d\tau [f_{1,k}(\tau)f_{2,m}(\tau) + f_{2,k}(\tau)f_{1,m}(\tau) + 4f_{3,k}(\tau)f_{3,m}(\tau)] \quad (3.66)$$

$$\sigma_{km}(t) = \frac{1}{16} \int_0^1 d\tau f_{4,k}(\tau)f_{4,m}(\tau). \quad (3.67)$$



### 3.4 Calculation of the Coefficient Functions

The remaining coefficient functions  $\varphi_k(t)$ ,  $\rho_{km}(t)$  and  $\sigma_{km}(t)$  are analytic functions of  $t$  at  $t = 0$ . To compute them we use the following integrals

$$\int_0^1 d\tau \cosh[(1-2\tau)x] = \frac{\sinh x}{x}, \quad (3.68)$$

$$\int_0^1 d\tau \sinh[(1-2\tau)x] = 0. \quad (3.69)$$

By differentiating these integrals with respect to  $x$  we obtain all other integrals we need

$$\int_0^1 d\tau \tau \cosh[(1-2\tau)x] = \frac{1}{2} \frac{\sinh x}{x}, \quad (3.70)$$

$$\int_0^1 d\tau \tau^2 \cosh[(1-2\tau)x] = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x^3} \right) \sinh x - \frac{1}{2} \frac{1}{x^2} \cosh x, \quad (3.71)$$

$$\int_0^1 d\tau \tau \sinh[(1-2\tau)x] = -\frac{1}{2} \frac{\cosh x}{x} + \frac{1}{2} \frac{\sinh x}{x^2}. \quad (3.72)$$

We also have the integrals

$$\int_0^1 d\tau \cosh[(1-2\tau)x] \cosh[(1-2\tau)y] = \frac{1}{2} \left\{ \frac{\sinh(x+y)}{x+y} + \frac{\sinh(x-y)}{x-y} \right\}, \quad (3.73)$$

$$\int_0^1 d\tau \cosh[(1-2\tau)x] \sinh[(1-2\tau)y] = 0, \quad (3.74)$$

$$\int_0^1 d\tau \sinh[(1-2\tau)x] \sinh[(1-2\tau)y] = \frac{1}{2} \left\{ \frac{\sinh(x+y)}{x+y} - \frac{\sinh(x-y)}{x-y} \right\}. \quad (3.75)$$

By using these integrals we obtain

$$\varphi_k(t) = \frac{1}{6} + \frac{3}{8} \frac{1}{(tB_k)^2} - \frac{1}{24} \coth(tB_k) \left( tB_k + 9 \frac{1}{(tB_k)} \right), \quad (3.76)$$

$$\begin{aligned} \sigma_{km}(t) = & \frac{1}{16} \coth(tB_k) \coth(tB_m) - \frac{1}{16} \frac{\coth(tB_k)}{tB_m} - \frac{1}{16} \frac{\coth(tB_m)}{tB_k} \\ & + \frac{1}{32} \frac{\coth(tB_m) + \coth(tB_k)}{t(B_k + B_m)} + \frac{1}{32} \frac{\coth(tB_m) - \coth(tB_k)}{t(B_k - B_m)}, \end{aligned} \quad (3.77)$$

$$\begin{aligned} \rho_{km}(t) = & -\frac{1}{48} \left\{ 4 + 9 \frac{1}{(tB_k)^2} + 9 \frac{1}{(tB_m)^2} - 8 \frac{1}{tB_k} \coth(tB_k) - 8 \frac{1}{tB_m} \coth(tB_m) \right. \\ & - (tB_k) \coth(tB_k) - (tB_m) \coth(tB_m) - 3 \frac{B_k}{tB_m^2} \coth(tB_k) \\ & - 3 \frac{B_m}{tB_k^2} \coth(tB_m) + 3 \left( \frac{B_k}{B_m} + \frac{B_m}{B_k} \right) \coth(tB_m) \coth(tB_k) \\ & - \frac{1}{2} \left[ \frac{B_m}{B_k} + \frac{B_k}{B_m} - 4 \right] \frac{\coth(tB_m) + \coth(tB_k)}{t(B_k + B_m)} \\ & \left. - \frac{1}{2} \left[ \frac{B_k}{B_m} + \frac{B_m}{B_k} + 4 \right] \frac{\coth(tB_m) - \coth(tB_k)}{t(B_k - B_m)} \right\}. \end{aligned} \quad (3.78)$$

### 3.5 Trace of the Heat Kernel Diagonal

The trace of the heat kernel diagonal in the general case within the considered approximation is given by

$$\Theta(t) \sim (4\pi t)^{-n/2} \Phi(t) \{1 + tB_1(t) + \dots\}, \quad (3.79)$$

where the function  $\Phi(t)$  was computed above and the coefficient  $B_1$  is given by

$$\begin{aligned} B_1(t) = & \left( \frac{1}{6} - \xi \right) R + \sum_{k=1}^N \left\{ \Psi^{\mu\alpha}(t) f_{5,k}(t) i E_k^{\gamma\beta} R_{\mu\alpha\nu\beta} + \varphi_k(t) \Pi_k^{\mu\nu} R_{\mu\nu} \right\} \\ & + \sum_{k=1}^N \sum_{m=1}^N \left\{ \rho_{km}(t) \Pi_k^{\mu\nu} \Pi_m^{\alpha\beta} R_{\mu\alpha\nu\beta} - \sigma_{km}(t) E_k^{\mu\alpha} E_m^{\nu\beta} R_{\mu\alpha\nu\beta} \right\}. \end{aligned} \quad (3.80)$$

Let us specify it for the two cases of interest.

### 3.5.1 Scalar Fields

For scalar fields we have

$$\Phi^{\text{scalar}}(t) = \prod_{k=1}^N \frac{tB_k}{\sinh(tB_k)}, \quad (3.81)$$

$$\begin{aligned} B_1^{\text{scalar}}(t) &= \left(\frac{1}{6} - \xi\right)R + \sum_{k=1}^N \varphi_k(t) \Pi_k^{\mu\nu} R_{\mu\nu} \\ &+ \sum_{k=1}^N \sum_{m=1}^N \left\{ \rho_{km}(t) \Pi_k^{\mu\nu} \Pi_m^{\alpha\beta} R_{\mu\alpha\nu\beta} - \sigma_{km}(t) E_k^{\mu\alpha} E_m^{\nu\beta} R_{\mu\alpha\nu\beta} \right\}. \end{aligned} \quad (3.82)$$

### 3.5.2 Spinor Fields

For the spinor fields we obtain

$$\Phi^{\text{spinor}}(t) = 2^{[n/2]} \prod_{k=1}^N tB_k \coth(tB_k), \quad (3.83)$$

$$\begin{aligned} B_1^{\text{spinor}}(t) &= -\frac{1}{12}R + \sum_{k=1}^N \varphi_k(t) \Pi_k^{\mu\nu} R_{\mu\nu} \\ &+ \sum_{k=1}^N \sum_{m=1}^N \left\{ \rho_{km}(t) \Pi_k^{\mu\nu} \Pi_m^{\alpha\beta} R_{\mu\alpha\nu\beta} - \lambda_{km}(t) E_k^{\mu\alpha} E_m^{\nu\beta} R_{\mu\alpha\nu\beta} \right\}, \end{aligned} \quad (3.84)$$

where

$$\begin{aligned} \lambda_{km}(t) &= \sigma_{km}(t) + \frac{1}{8} \frac{\tanh(tB_m)}{tB_k} + \frac{1}{8} \frac{\tanh(tB_k)}{tB_m} \\ &- \frac{1}{8} \tanh(tB_m) \coth(tB_k) - \frac{1}{8} \tanh(tB_k) \coth(tB_m). \end{aligned} \quad (3.85)$$

## 3.6 Equal Magnetic Fields

We will specify the obtained result for the case when all magnetic invariants are equal to each other, that is,

$$B_1 = \dots = B_N = B. \quad (3.86)$$

### 3.6.1 Scalar Fields

For scalar fields it takes the form

$$\Phi^{\text{scalar}}(t) = \left( \frac{tB}{\sinh(tB)} \right)^N, \quad (3.87)$$

$$\begin{aligned} B_1^{\text{scalar}}(t) = & \left( \frac{1}{6} - \xi \right) R + \varphi(t) H_1^{\mu\nu} R_{\mu\nu} \\ & + \rho(t) H_1^{\mu\nu} H_1^{\alpha\beta} R_{\mu\alpha\nu\beta} - \sigma(t) X_1^{\mu\nu} X_1^{\alpha\beta} R_{\mu\alpha\nu\beta}, \end{aligned} \quad (3.88)$$

where

$$H_1^{\mu\nu} = \sum_{k=1}^N \Pi_k^{\mu\nu}, \quad X_1^{\mu\nu} = \sum_{k=1}^N E_k^{\mu\nu}, \quad (3.89)$$

$$\varphi(t) = \frac{1}{6} + \frac{3}{8} \frac{1}{(tB)^2} - \frac{1}{24} tB \coth(tB) - \frac{3}{8} \frac{\coth(tB)}{tB}, \quad (3.90)$$

$$\sigma(t) = \frac{1}{16} - \frac{3}{32} \frac{\coth(tB)}{tB} + \frac{3}{32} \frac{1}{\sinh^2(tB)} \quad (3.91)$$

$$\rho(t) = -\frac{5}{24} - \frac{3}{8} \frac{1}{(tB)^2} + \frac{1}{24} tB \coth(tB) + \frac{7}{16} \frac{\coth(tB)}{tB} - \frac{1}{16} \frac{1}{\sinh^2(tB)} \quad (3.92)$$

### 3.6.2 Spinor Fields

For the spinor fields we obtain

$$\Phi^{\text{spinor}}(t) = 2^{[n/2]} [tB \coth(tB)]^N, \quad (3.93)$$

$$\begin{aligned} B_1^{\text{spinor}}(t) = & -\frac{1}{12} R + \varphi(t) H_1^{\mu\nu} R_{\mu\nu} \\ & + \rho(t) H_1^{\mu\nu} H_1^{\alpha\beta} R_{\mu\alpha\nu\beta} - \lambda(t) X_1^{\mu\nu} X_1^{\alpha\beta} R_{\mu\alpha\nu\beta}, \end{aligned} \quad (3.94)$$

where

$$\lambda(t) = -\frac{3}{16} + \frac{3}{32} \frac{1}{\sinh^2(tB)} + \frac{1}{4} \frac{\tanh(tB)}{tB} - \frac{3}{32} \frac{\coth(tB)}{tB}. \quad (3.95)$$

### 3.7 Electric and Magnetic Fields

Now we specify the above results for the pseudo-Euclidean case when there is one electric field and  $(N - 1)$  equal magnetic fields. By using the recipe (3.27) we obtain the following results.

#### 3.7.1 Scalar Fields

For scalar fields we have

$$\Phi^{\text{scalar}}(t) = \frac{tE}{\sin(tE)} \left( \frac{tB}{\sinh(tB)} \right)^{N-1}, \quad (3.96)$$

$$\begin{aligned} B_1^{\text{scalar}}(t) = & \left( \frac{1}{6} - \xi \right) R - \tilde{\varphi}(t) \Pi_1^{\mu\nu} R_{\mu\nu} + \varphi(t) H_2^{\mu\nu} R_{\mu\nu} + \tilde{\rho}(t) \Pi_1^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\alpha\nu\beta} \\ & + \tilde{\sigma}(t) E_1^{\mu\alpha} E_1^{\nu\beta} R_{\mu\alpha\nu\beta} - 2\rho_1(t) H_2^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\alpha\nu\beta} + 2\sigma_1(t) X_2^{\nu\alpha} E_1^{\nu\beta} R_{\mu\alpha\nu\beta} \\ & + \rho(t) H_2^{\mu\nu} H_2^{\alpha\beta} R_{\mu\alpha\nu\beta} - \sigma(t) X_2^{\mu\alpha} X_2^{\nu\beta} R_{\mu\alpha\nu\beta}, \end{aligned} \quad (3.97)$$

where

$$H_2^{\mu\nu} = \sum_{k=2}^N \Pi_k^{\mu\nu}, \quad X_2^{\mu\nu} = \sum_{k=2}^N E_k^{\mu\nu}, \quad (3.98)$$

$$\tilde{\varphi}(t) = \frac{1}{6} - \frac{3}{8} \frac{1}{(tE)^2} - \frac{1}{24} tE \cot(tE) + \frac{3}{8} \frac{\cot(tE)}{tE}, \quad (3.99)$$

$$\tilde{\rho}(t) = -\frac{5}{24} + \frac{3}{8} \frac{1}{(tE)^2} + \frac{1}{24} tE \cot(tE) - \frac{7}{16} \frac{\cot(tE)}{tE} + \frac{1}{16} \frac{1}{\sin^2(tE)}, \quad (3.100)$$

$$\tilde{\sigma}(t) = \frac{1}{16} + \frac{3}{32} \frac{\cot(tE)}{tE} - \frac{3}{32} \frac{1}{\sin^2(tE)}, \quad (3.101)$$

$$\begin{aligned} \sigma_1(t) = & \frac{1}{16} \cot(tE) \coth(tB) - \frac{1}{16} \frac{\cot(tE)}{tB} - \frac{1}{16} \frac{\coth(tB)}{tE} \\ & + \frac{1}{16} \frac{B \cot(tE) + E \coth(tB)}{t(B^2 + E^2)}. \end{aligned} \quad (3.102)$$

$$\begin{aligned}
 \rho_1(t) = & -\frac{1}{48} \left\{ 4 - 9 \frac{1}{(tE)^2} + 9 \frac{1}{(tB)^2} + 8 \frac{1}{tE} \cot(tE) - 8 \frac{1}{tB} \coth(tB) \right. \\
 & - (tE) \cot(tE) - (tB) \coth(tB) \\
 & - 3 \frac{E}{tB^2} \cot(tE) + 3 \frac{B}{tE^2} \coth(tB) + 3 \left( \frac{E}{B} - \frac{B}{E} \right) \coth(tB) \cot(tE) \\
 & \left. + \frac{5B^2 - E^2}{tB(B^2 + E^2)} \coth(tB) - \frac{5E^2 - B^2}{tE(B^2 + E^2)} \cot(tE) \right\}. \quad (3.103)
 \end{aligned}$$

### 3.7.2 Spinor Fields

For the spinor fields we obtain

$$\Phi^{\text{spinor}}(t) = 2^{[n/2]} tE \cot(tE) [tB \coth(tB)]^{N-1}, \quad (3.104)$$

$$\begin{aligned}
 B_1^{\text{spinor}}(t) = & -\frac{1}{12} R - \tilde{\varphi}(t) \Pi_1^{\mu\nu} R_{\mu\nu} + \varphi(t) H_2^{\mu\nu} R_{\mu\nu} + \tilde{\rho}(t) \Pi_1^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\alpha\nu\beta} \\
 & + \tilde{\lambda}(t) E_1^{\mu\alpha} E_1^{\nu\beta} R_{\mu\alpha\nu\beta} - 2\rho_1(t) H_2^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\alpha\nu\beta} + 2\lambda_1(t) X_2^{\mu\alpha} E_1^{\nu\beta} R_{\mu\alpha\nu\beta} \\
 & + \rho(t) H_2^{\mu\nu} H_2^{\alpha\beta} R_{\mu\alpha\nu\beta} - \lambda(t) X_2^{\mu\alpha} X_2^{\nu\beta} R_{\mu\alpha\nu\beta}, \quad (3.105)
 \end{aligned}$$

where

$$\tilde{\lambda}(t) = -\frac{3}{16} - \frac{3}{32} \frac{1}{\sin^2(tE)} + \frac{1}{4} \frac{\tan(tE)}{tE} + \frac{3}{32} \frac{\cot(tE)}{tE}, \quad (3.106)$$

$$\begin{aligned}
 \lambda_1(t) = & \frac{1}{16} \cot(tE) \coth(tB) - \frac{1}{16} \frac{\cot(tE)}{tB} - \frac{1}{16} \frac{\coth(tB)}{tE} \\
 & + \frac{1}{16} \frac{B \cot(tE) + E \coth(tB)}{t(B^2 + E^2)} + \frac{1}{8} \frac{\tanh(tB)}{tE} - \frac{1}{8} \frac{\tan(tE)}{tB} \\
 & - \frac{1}{8} \tanh(tB) \cot(tE) + \frac{1}{8} \tan(tE) \coth(tB). \quad (3.107)
 \end{aligned}$$

## 4 Imaginary Part of the Effective Lagrangian

Now, we can compute the imaginary part of the effective Lagrangian in the same approximation taking into account linear terms in the curvature. The effective action is given by the integral over  $t$  of the trace of the heat kernel diagonal. Of

course, it should be properly regularized as discussed above. The most important point we want to make is that in the presence of the electric field the heat kernel is no longer a nice analytic function of  $t$  but it becomes a meromorphic function of  $t$  in the complex plane of  $t$  with poles on the real axis determined by the trigonometric functions in the coefficient functions computed above. As was pointed out first by Schwinger these poles should be carefully avoided by deforming the contour of integration which leads to an imaginary part of the effective action determined by the contribution of the residues of the poles. This imaginary part is always finite and does not depend on the regularization. We compute below the imaginary part of the effective Lagrangian for the scalar and the spinor fields.

The trace of the heat kernel  $\Theta(t)$  was computed above and is given by (3.79). Now, by using (2.32) the calculation of the imaginary part of the effective Lagrangian is reduced to the calculation of the residues of the functions  $t^{-n/2-1}e^{-tm^2}\Phi(t)$  and  $t^{-n/2}e^{-tm^2}\Phi(t)B_1(t)$  at the poles on the real line. By using the result (3.96) and (3.104) for the function  $\Phi$  it is not difficult to see that the function  $t^{-n/2-1}e^{-tm^2}\Phi(t)$  is a meromorphic function with isolated simple poles at  $t_k = k\pi/E$  with  $k = 1, 2, \dots$ . The function  $t^{-n/2}e^{-tm^2}\Phi(t)B_1(t)$  is also a meromorphic function with the same poles but the poles could be double or even triple. The imaginary part is, then, simply evaluated by summing the residues of the integrand at the poles. It has the following form

$$\begin{aligned} \text{Im } \mathcal{L} = & \pi(4\pi)^{-n/2}E^{n/2}G_0(x, y) + \pi(4\pi)^{-n/2}E^{n/2-1}\left[G_1(x, y)R \right. \\ & + G_2(x, y)\Pi_1^{\mu\nu}R_{\mu\nu} + G_3(x, y)H_2^{\mu\nu}R_{\mu\nu} + G_4(x, y)\Pi_1^{\mu\nu}\Pi_1^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ & + G_5(x, y)E_1^{\mu\alpha}E_1^{\nu\beta}R_{\mu\alpha\nu\beta} + G_6(x, y)H_2^{\mu\nu}\Pi_1^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ & + G_7(x, y)X_2^{\mu\alpha}E_1^{\nu\beta}R_{\mu\alpha\nu\beta} + G_8(x, y)H_2^{\mu\nu}H_2^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ & \left. + G_9(x, y)X_2^{\mu\alpha}X_2^{\nu\beta}R_{\mu\alpha\nu\beta}\right], \end{aligned} \quad (4.1)$$

where

$$x = \frac{B}{E}, \quad y = \frac{m^2}{E}, \quad (4.2)$$

and  $G_i(x, y)$  are some functions computed below.

## 4.1 Scalar Fields

At this point it is useful to introduce some auxiliary functions so that the final result for the quantities  $G_i^{\text{scalar}}(x, y)$  can be written in a somewhat compact form,

namely

$$f_k(x, y) = \left[ \frac{k\pi x}{\sinh(k\pi x)} \right]^{N-1} \exp(-k\pi y) , \quad (4.3)$$

$$g_k(x, y) = (N-1)(k\pi x) \coth(k\pi x) + k\pi y , \quad (4.4)$$

$$\begin{aligned} h_k(x, y) &= \frac{1}{2}N(N-1)(k\pi x)^2 \coth^2(k\pi x) + \left(\frac{n}{2} - N\right)k\pi y \\ &+ \frac{1}{2}(N-1)[(n-2N) + 2k\pi y](k\pi x) \coth(k\pi x) \\ &+ \frac{1}{2}(k\pi)^2[1 - (N-1)x^2 + y^2] . \end{aligned} \quad (4.5)$$

$$l_k(x, y) = -k\pi x + \left[\left(\frac{n}{2} - N\right) + k\pi y\right] \coth(k\pi x) + N(k\pi x) \coth^2(k\pi x) , \quad (4.6)$$

$$\begin{aligned} \Omega_{1,k}(x, y) &= \frac{1}{8} + \frac{n-2N}{48} - \frac{3}{8(k\pi)^2} \left(\frac{n}{2} - N + 2\right) \\ &+ \frac{1}{24} \left(1 - \frac{9}{(k\pi)^2}\right) g_k(x, y) , \end{aligned} \quad (4.7)$$

$$\begin{aligned} \Omega_{2,k}(x, y) &= -\frac{1}{6} - \frac{n-2N}{48} + \frac{1}{32(k\pi)^2} \left(\frac{n}{2} - N + 2\right) \left(\frac{n}{2} - N + 13\right) \\ &- \frac{1}{24} \left(1 - \frac{21}{2(k\pi)^2}\right) g_k(x, y) + \frac{1}{16} \frac{h_k(x, y)}{(k\pi)^2} , \end{aligned} \quad (4.8)$$

$$\begin{aligned} \Omega_{3,k}(x, y) &= \frac{1}{16} - \frac{3}{64(k\pi)^2} \left(\frac{n}{2} - N + 1\right) \left(\frac{n}{2} - N + 2\right) \\ &- \frac{3}{32(k\pi)^2} [h_k(x, y) + g_k(x, y)] , \end{aligned} \quad (4.9)$$

$$\begin{aligned} \Omega_{4,k}(x, y) &= -\frac{1}{8} - \frac{n-2N}{48} + \frac{3}{8(k\pi)^2} \left(1 - \frac{1}{x^2}\right) + \frac{1}{8} \left(\frac{1}{x} - x\right) \frac{l_k(x, y)}{k\pi} \\ &+ \frac{1}{8(k\pi)^2} \left[\left(\frac{n}{2} - N + 1\right) + g_k(x, y)\right] \left[\frac{3x^4 - 1}{x^2(x^2 + 1)}\right] \\ &- \frac{1}{8} \frac{\coth(k\pi x)}{k\pi x} \left[\frac{x^4 - 3}{x^2 + 1} - \frac{(k\pi x)^2}{3}\right] - \frac{1}{24} g_k(x, y) , \end{aligned} \quad (4.10)$$

$$\begin{aligned} \Omega_{5,k}(x, y) &= \frac{1}{8(k\pi)^2} \left[\left(\frac{n}{2} - N + 1\right) + g_k(x, y)\right] \frac{1}{x(x^2 + 1)} - \frac{1}{8} \frac{l_k(x, y)}{(k\pi)} \\ &- \frac{1}{8} \frac{\coth(k\pi x)}{k\pi x} \left(\frac{x^3}{x^2 + 1}\right) . \end{aligned} \quad (4.11)$$



By using these quantities we obtain the functions  $G_i^{\text{scalar}}(x, y)$  in the form of the following series

$$G_0^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2}} f_k(x, y) , \quad (4.12)$$

$$G_1^{\text{scalar}}(x, y) = \left( \frac{1}{6} - \xi \right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) , \quad (4.13)$$

$$G_2^{\text{scalar}}(x, y) = - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{1,k}(x, y) , \quad (4.14)$$

$$G_3^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \varphi(k\pi x) , \quad (4.15)$$

$$G_4^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{2,k}(x, y) , \quad (4.16)$$

$$G_5^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{3,k}(x, y) , \quad (4.17)$$

$$G_6^{\text{scalar}}(x, y) = - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{4,k}(x, y) , \quad (4.18)$$

$$G_7^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{5,k}(x, y) , \quad (4.19)$$

$$G_8^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \rho(k\pi x) , \quad (4.20)$$

$$G_9^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \sigma(k\pi x) . \quad (4.21)$$

## 4.2 Spinor Fields

Exactly as we did in the previous section, we introduce, now, some auxiliary functions that will be useful in the presentation of the final result, namely

$$f_{S,k}(x, y) = \left[ (k\pi x) \coth(k\pi x) \right]^{N-1} \exp(-k\pi y) , \quad (4.22)$$

$$g_{S,k}(x, y) = (N-1)(k\pi x) \coth(k\pi x) - (N-1)(k\pi x) \tanh(k\pi x) + k\pi y , \quad (4.23)$$

$$\begin{aligned} h_{S,k}(x, y) &= \frac{1}{2}(k\pi y)^2 - (N-1)^2(k\pi x)^2 + \left(\frac{n}{2} - N\right)k\pi y \\ &+ \frac{1}{2}(N-1)(n-2N+2k\pi y)(k\pi x) \left[ \coth(k\pi x) - \tanh(k\pi x) \right] \\ &+ \frac{1}{2}N(N-1)(k\pi x)^2 \coth^2(k\pi x) + \frac{1}{2}(N-1)(N-2)(k\pi x)^2 \tanh^2(k\pi x) , \end{aligned} \quad (4.24)$$

$$l_{S,k}(x, y) = -Nk\pi x + \left(\frac{n}{2} - N + k\pi y\right) \coth(k\pi x) + N(k\pi x) \coth^2(k\pi x) , \quad (4.25)$$

$$p_{S,k}(x, y) = (N-2)k\pi x + \left(\frac{n}{2} - N + k\pi y\right) \tanh(k\pi x) - (N-2)(k\pi x) \tanh^2(k\pi x) , \quad (4.26)$$

$$\Lambda_{1,k}(x, y) = \frac{1}{8} + \frac{n-2N}{48} - \frac{3}{8(k\pi)^2} \left(\frac{n}{2} - N + 2\right) + \frac{1}{24} \left(1 - \frac{9}{(k\pi)^2}\right) g_{S,k}(x, y) , \quad (4.27)$$

$$\begin{aligned} \Lambda_{2,k}(x, y) &= -\frac{1}{6} - \frac{n-2N}{48} + \frac{1}{32(k\pi)^2} \left(\frac{n}{2} - N + 2\right) \left(\frac{n}{2} - N + 13\right) \\ &- \frac{1}{24} \left(1 - \frac{21}{2(k\pi)^2}\right) g_{S,k}(x, y) + \frac{1}{16} \frac{h_{S,k}(x, y)}{(k\pi)^2} , \end{aligned} \quad (4.28)$$

$$\begin{aligned} \Lambda_{3,k}(x, y) &= -\frac{3}{16} - \frac{3}{64(k\pi)^2} \left(\frac{n}{2} - N + 1\right) \left(\frac{n}{2} - N + 2\right) \\ &- \frac{3}{32(k\pi)^2} \left[ g_{S,k}(x, y) + h_{S,k}(x, y) \right] , \end{aligned} \quad (4.29)$$

$$\begin{aligned}
 \Lambda_{4,k}(x, y) &= -\frac{1}{8} - \frac{n-2N}{48} + \frac{3}{8(k\pi)^2} \left(1 - \frac{1}{x^2}\right) + \frac{1}{8} \left(\frac{1}{x} - x\right) \frac{l_{S,k}(x, y)}{k\pi} \\
 &+ \frac{1}{8(k\pi)^2} \left(\frac{n}{2} - N + 1 + g_{S,k}(x, y)\right) \frac{3x^4 - 1}{x^2(x^2 + 1)} \\
 &- \frac{1}{8} \frac{\coth(k\pi x)}{k\pi x} \left[ \frac{x^4 - 3}{x^2 + 1} - \frac{(k\pi x)^2}{3} \right] - \frac{1}{24} g_{S,k}(x, y), \quad (4.30)
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_{5,k}(x, y) &= \frac{1}{8(k\pi)^2} \left(\frac{n}{2} - N + 1 + g_{S,k}(x, y)\right) \frac{1}{x(x^2 + 1)} - \frac{1}{8} \frac{l_{S,k}(x, y)}{k\pi} \\
 &- \frac{1}{8} \frac{\coth(k\pi x)}{k\pi x} \left(\frac{x^3}{x^2 + 1}\right) + \frac{1}{4(k\pi)} \left[ \tanh(k\pi x) + p_{S,k}(x, y) \right]. \quad (4.31)
 \end{aligned}$$

By using the above functions we can write the explicit expression for the quantities  $G_i^{\text{spinor}}(x, y)$

$$G_0^{\text{spinor}}(x, y) = 2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2}} f_{S,k}(x, y), \quad (4.32)$$

$$G_1^{\text{spinor}}(x, y) = -\frac{2^{[\frac{n}{2}]} }{12} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y), \quad (4.33)$$

$$G_2^{\text{spinor}}(x, y) = -2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{1,k}(x, y), \quad (4.34)$$

$$G_3^{\text{spinor}}(x, y) = 2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \varphi(k\pi x), \quad (4.35)$$

$$G_4^{\text{spinor}}(x, y) = 2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{2,k}(x, y), \quad (4.36)$$

$$G_5^{\text{spinor}}(x, y) = 2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{3,k}(x, y), \quad (4.37)$$

$$G_6^{\text{spinor}}(x, y) = -2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{4,k}(x, y), \quad (4.38)$$

$$G_7^{\text{spinor}}(x, y) = 2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{5,k}(x, y) , \quad (4.39)$$

$$G_8^{\text{spinor}}(x, y) = 2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \rho(k\pi x) , \quad (4.40)$$

$$G_9^{\text{spinor}}(x, y) = -2^{[\frac{n}{2}]} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \lambda(k\pi x) . \quad (4.41)$$

Notice that because of the infrared cutoff factor  $e^{-k\pi y}$  the functions  $G_i(x, y)$  are exponentially small for massive fields in weak electric fields when the parameter is large,  $y \gg 1$  (that is,  $m^2 \gg E$ ), independently on  $x$ . In this case, all these functions are approximated by just the first term of the series corresponding to  $k = 1$ .

## 5 Strong Electric Field in Four Dimensions

The formulas obtained in the previous section are very general and are valid in any dimensions. In this section we will present some particular cases of major interest.

### 5.1 Four Dimensions

In this section we will consider the physical case when  $n = 4$ . Obviously in four dimensions we only have two invariants, and, therefore,  $N = 2$ . The imaginary part of the effective Lagrangian reads now

$$\begin{aligned} \text{Im } \mathcal{L} = & \pi(4\pi)^{-2} E^2 G_0(x, y) + \pi(4\pi)^{-2} E \left[ G_1(x, y) R \right. \\ & + G_2(x, y) \Pi_1^{\mu\nu} R_{\mu\nu} + G_3(x, y) \Pi_2^{\mu\nu} R_{\mu\nu} + G_4(x, y) \Pi_1^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\alpha\nu\beta} \\ & + G_5(x, y) E_1^{\mu\alpha} E_1^{\nu\beta} R_{\mu\alpha\nu\beta} + G_6(x, y) \Pi_2^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\alpha\nu\beta} \\ & + G_7(x, y) E_2^{\mu\alpha} E_1^{\nu\beta} R_{\mu\alpha\nu\beta} + G_8(x, y) \Pi_2^{\mu\nu} \Pi_2^{\alpha\beta} R_{\mu\alpha\nu\beta} \\ & \left. + G_9(x, y) E_2^{\mu\alpha} E_2^{\nu\beta} R_{\mu\alpha\nu\beta} \right] . \end{aligned} \quad (5.42)$$

For scalar fields in four dimensions the functions  $G_i^{\text{scalar}}(x, y)$  take the form

$$G_0^{\text{scalar}}(x, y) = \frac{x}{\pi} \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{k \sinh(k\pi x)}, \quad (5.43)$$

$$G_1^{\text{scalar}}(x, y) = \left( \frac{1}{6} - \xi \right) x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)}, \quad (5.44)$$

$$G_2^{\text{scalar}}(x, y) = -x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{8} - \frac{3}{4(k\pi)^2} + \frac{1}{24} \left( k\pi - \frac{9}{k\pi} \right) [y + x \coth(k\pi x)] \right\}, \quad (5.45)$$

$$G_3^{\text{scalar}}(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{6} + \frac{3}{8} \frac{1}{(k\pi x)^2} - \frac{1}{24} k\pi x \coth(k\pi x) - \frac{3}{8} \frac{\coth(k\pi x)}{k\pi x} \right\}, \quad (5.46)$$

$$G_4^{\text{scalar}}(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{13}{96} + \frac{13}{16(k\pi)^2} - \frac{x^2}{32} + \frac{y^2}{32} + \left( \frac{7}{16(k\pi)} - \frac{k\pi}{24} \right) y \right. \\ \left. + \left( \frac{y}{16} + \frac{7}{16k\pi} - \frac{k\pi}{24} \right) x \coth(k\pi x) + \frac{x^2}{16} \coth^2(k\pi x) \right\}, \quad (5.47)$$

$$G_5^{\text{scalar}}(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{64} - \frac{3}{32(k\pi)^2} + \frac{3x^2}{64} - \frac{3y^2}{64} - \frac{3}{32k\pi} y \right. \\ \left. - \frac{3}{32} \left( y + \frac{1}{k\pi} \right) \coth(k\pi x) - \frac{3}{32} x^2 \coth^2(k\pi x) \right\}, \quad (5.48)$$

$$G_6^{\text{scalar}}(x, y) = -\frac{x}{x^2 + 1} \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{1}{4} - \frac{1}{2(k\pi x)^2} + \frac{3x^2}{4(k\pi)^2} + \frac{x^2}{8} (x^2 - 1) \right. \\ - \frac{k\pi}{24} y (x^2 + 1) - \frac{1}{8k\pi} y \left( \frac{1}{x^2} - 3x^2 \right) - \frac{1}{4} (x^4 - 1) \coth^2(k\pi x) \\ \left. + \left[ \frac{1}{4k\pi} \left( x^3 + \frac{1}{x} \right) + \frac{y}{8} \left( \frac{1}{x} - x^3 \right) \right] \coth(k\pi x) \right\}, \quad (5.49)$$

$$\begin{aligned}
 G_7^{\text{scalar}}(x, y) &= \frac{1}{x^2 + 1} \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{8(k\pi)^2} + \frac{x^2}{8}(x^2 + 1) + \frac{y}{8k\pi} \right. \\
 &\quad - \frac{x}{8} \left[ y(1 + x^2) - \frac{1}{k\pi}(1 - x^2) \right] \coth(k\pi x) \\
 &\quad \left. - \frac{1}{4}x^2(x^2 + 1) \coth^2(k\pi x) \right\}, \tag{5.50}
 \end{aligned}$$

$$\begin{aligned}
 G_8^{\text{scalar}}(x, y) &= x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{7}{48} - \frac{3}{8} \frac{1}{(k\pi x)^2} - \frac{1}{16} \coth^2(k\pi x) \right. \\
 &\quad \left. + \left( \frac{k\pi}{24}x + \frac{7}{16k\pi x} \right) \coth(k\pi x) \right\}, \tag{5.51}
 \end{aligned}$$

$$\begin{aligned}
 G_9^{\text{scalar}}(x, y) &= x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{1}{32} - \frac{3}{32k\pi x} \coth(k\pi x) + \frac{3}{32} \coth^2(k\pi x) \right\}. \tag{5.52}
 \end{aligned}$$

For spinor fields in four dimensions the functions  $G_i^{\text{spinor}}(x, y)$  take the form

$$G_0^{\text{spinor}}(x, y) = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \coth(k\pi x) e^{-k\pi y}, \tag{5.53}$$

$$G_1^{\text{spinor}}(x, y) = -\frac{x}{3} \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y}, \tag{5.54}$$

$$\begin{aligned}
 G_2^{\text{spinor}}(x, y) &= -4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ \frac{1}{8} - \frac{3}{4(k\pi)^2} \right. \\
 &\quad \left. + \frac{1}{24} \left( k\pi - \frac{9}{k\pi} \right) \left[ y + x \coth(k\pi x) - x \tanh(k\pi x) \right] \right\}, \tag{5.55}
 \end{aligned}$$

$$\begin{aligned}
 G_3^{\text{spinor}}(x, y) &= 4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ \frac{1}{6} + \frac{3}{8} \frac{1}{(k\pi x)^2} - \frac{1}{24} k\pi x \coth(k\pi x) \right. \\
 &\quad \left. - \frac{3}{8} \frac{\coth(k\pi x)}{k\pi x} \right\}, \tag{5.56}
 \end{aligned}$$

$$\begin{aligned}
 G_4^{\text{spinor}}(x, y) &= 4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ -\frac{1}{6} + \frac{13}{16(k\pi)^2} - \frac{x^2}{16} + \frac{y^2}{32} \right. \\
 &\quad - \frac{y}{24} \left( k\pi - \frac{21}{2k\pi} \right) + \frac{x^2}{16} \coth^2(k\pi x) \\
 &\quad \left. - \frac{x}{24} \left( k\pi - \frac{21}{2k\pi} - \frac{3y}{2} \right) \left[ \coth(k\pi x) - \tanh(k\pi x) \right] \right\}, \quad (5.57)
 \end{aligned}$$

$$\begin{aligned}
 G_5^{\text{spinor}}(x, y) &= 4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ -\frac{3}{16} - \frac{3}{32(k\pi)^2} + \frac{3x^2}{32} \right. \\
 &\quad - \frac{3y^2}{64} - \frac{3y}{32(k\pi)} - \frac{3x^2}{32} \coth^2(k\pi x) \\
 &\quad \left. - \frac{3x}{32} \left( \frac{1}{k\pi} + y \right) \left[ \coth(k\pi x) - \tanh(k\pi x) \right] \right\}, \quad (5.58)
 \end{aligned}$$

$$\begin{aligned}
 G_6^{\text{spinor}}(x, y) &= -\frac{4x}{x^2 + 1} \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ -\frac{3}{8} + \frac{3}{4(k\pi)^2} \left( x^2 - \frac{2}{3x^2} \right) \right. \\
 &\quad + \frac{x^2}{8} (2x^2 - 1) - \frac{x^2 y}{24} \left( k\pi - \frac{9}{k\pi} \right) - \frac{y}{24} \left( k\pi + \frac{3}{k\pi x^2} \right) \\
 &\quad + \left[ \frac{1}{4k\pi x} + \frac{y}{8x} - \frac{x^3}{8} \left( y - \frac{2}{k\pi} \right) \right] \coth(k\pi x) \\
 &\quad + \left[ \frac{1}{8k\pi x} + \frac{k\pi x}{24} + \frac{x^3}{24} \left( k\pi - \frac{9}{k\pi} \right) \right] \tanh(k\pi x) \\
 &\quad \left. - \frac{1}{4} (x^4 - 1) \coth^2(k\pi x) \right\}, \quad (5.59)
 \end{aligned}$$

$$\begin{aligned}
 G_7^{\text{spinor}}(x, y) &= \frac{4}{x^2 + 1} \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ \frac{1}{8(k\pi)^2} + \frac{x^2}{4} (x^2 + 1) \right. \\
 &\quad + \frac{y}{8k\pi} - \frac{x^2}{4} (x^2 + 1) \coth^2(k\pi x) \\
 &\quad + \frac{x}{8} \left[ \frac{1}{k\pi} (1 - x^2) - y(1 + x^2) \right] \coth(k\pi x) \\
 &\quad \left. + \frac{x}{8} \left[ \frac{1}{k\pi} (1 + 2x^2) + 2y(1 + x^2) \right] \tanh(k\pi x) \right\}, \quad (5.60)
 \end{aligned}$$

$$G_8^{\text{spinor}}(x, y) = 4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ -\frac{7}{48} - \frac{3}{8} \frac{1}{(k\pi x)^2} + \frac{1}{24} k\pi x \coth(k\pi x) + \frac{7}{16} \frac{\coth(k\pi x)}{k\pi x} - \frac{1}{16} \coth^2(k\pi x) \right\}, \quad (5.61)$$

$$G_9^{\text{spinor}}(x, y) = -4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ -\frac{9}{32} + \frac{3}{32} \coth^2(k\pi x) + \frac{1}{4} \frac{\tanh(k\pi x)}{k\pi x} - \frac{3}{32} \frac{\coth(k\pi x)}{k\pi x} \right\}, \quad (5.62)$$

## 5.2 Supercritical Electric Field

As we already mentioned above, the functions  $G_i(x, y)$  are exponentially small for massive fields in weak electric fields for large  $y = m^2/E$ , as  $y \rightarrow \infty$ . Now we are considering the opposite case of light (or massless) fields in strong (supercritical) electric fields, when  $y \rightarrow 0$  with a fixed  $x$ . This corresponds to the regime

$$m^2 \ll B, E. \quad (5.63)$$

### 5.2.1 Scalar Fields

The infrared (massless) limit for scalar fields is regular—there are no infrared divergences. This is due to the presence of the hyperbolic sine  $\sinh(k\pi x)$  in the denominator, which gives a cut-off for large  $k$  in the series, and therefore, assures its convergence. The result for the massless limit in the scalar case can be simply obtained by setting  $y = 0$  in the above formulas for the functions  $G_i(x, y)$ .

### 5.2.2 Spinor Fields

The spinor case is quite different. The presence of the hyperbolic cotangent  $\coth(k\pi x)$  does not provide a cut-off for the convergence of the series as  $k \rightarrow \infty$ . This leads, in the spinor case in four dimensions, to the presence of infrared divergences as  $y = m^2/E \rightarrow 0$ . By carefully studying the behavior of the series as  $k \rightarrow \infty$  for a finite  $y$  and then letting  $y \rightarrow 0$  we compute the asymptotic expansion of the functions  $G_i^{\text{spinor}}(x, y)$  as  $y \rightarrow 0$ .



We obtain

$$G_0^{\text{spinor}}(x, y) = \frac{2x}{3} + O(y), \quad (5.64)$$

$$G_1^{\text{spinor}}(x, y) = -\frac{1}{3\pi} \frac{x}{y} + \frac{x}{8} + O(y), \quad (5.65)$$

$$G_2^{\text{spinor}}(x, y) = -\frac{2}{3\pi} \frac{x}{y} + \frac{3x}{4} + O(y), \quad (5.66)$$

$$G_3^{\text{spinor}}(x, y) = -\frac{1}{6\pi} \frac{x^2}{y^2} + \frac{2}{3\pi} \frac{x}{y} + \frac{3}{2\pi} \log(\pi y) + \frac{x^2(\pi x - 24) + 18}{72x} + O(y), \quad (5.67)$$

$$G_4^{\text{spinor}}(x, y) = -\frac{5}{6\pi} \frac{x}{y} + \frac{7x}{8} + O(y), \quad (5.68)$$

$$G_5^{\text{spinor}}(x, y) = -\frac{3}{4\pi} \frac{x}{y} + \frac{5x}{16} + O(y), \quad (5.69)$$

$$G_6^{\text{spinor}}(x, y) = \frac{x^2 - 1}{6\pi(x^2 + 1)} \frac{x^2}{y^2} + \frac{2}{3\pi} \frac{x}{y} - \frac{x^4 - 3}{2\pi(x^2 + 1)} \log(\pi y) - \frac{6\pi(9x^4 + 3x^2 - 4) - 36x(x^4 - 1) + \pi^2 x^3(x^2 - 1)}{72\pi x(x^2 + 1)} + O(y), \quad (5.70)$$

$$G_7^{\text{spinor}}(x, y) = -\frac{x(x^2 + 2)}{2\pi(x^2 + 1)} \log(\pi y) + \frac{6x(x^2 + 1) + \pi}{12\pi(x^2 + 1)} + O(y), \quad (5.71)$$

$$G_8^{\text{spinor}}(x, y) = \frac{1}{6\pi} \frac{x^2}{y^2} - \frac{5}{6\pi} \frac{x}{y} - \frac{7}{4\pi} \log(\pi y) - \frac{x^2(\pi x - 30) + 18}{72x} + O(y), \quad (5.72)$$

$$G_9^{\text{spinor}}(x, y) = \frac{3}{4\pi} \frac{x}{y} + \frac{5}{8\pi} \log(\pi y) - \frac{3x}{8} + O(y). \quad (5.73)$$

Thus, we clearly see the infrared divergences of order  $x^2/y^2 = B^2/m^4$ ,  $x/y = B/m^2$  and  $\log y = \log(m^2/E)$ .

### 5.3 Pure Electric Field

We analyze now the case of pure electric field without a magnetic field, that is,  $B = 0$ , which corresponds to the limit  $x \rightarrow 0$  with fixed  $y$ . This corresponds to the

physical regime when

$$B \ll m^2, E. \quad (5.74)$$

In this discussion we present the results in arbitrary dimension first and then we specialize them to the physical dimension  $n = 4$ .

### 5.3.1 Scalar Fields

We now evaluate the functions  $G_i(x, y)$  for  $x = 0$  and a finite  $y$ . In this limit we are presented with series of the following general form

$$\chi_n^{\text{scalar}}(y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{-k\pi y}}{k^{n/2}}. \quad (5.75)$$

This series can be expressed in terms of the polylogarithmic function defined by

$$\text{Li}_j(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^j}, \quad (5.76)$$

so that, we have

$$\chi_n^{\text{scalar}}(y) = -\text{Li}_{\frac{n}{2}}(-e^{-\pi y}). \quad (5.77)$$

It is not difficult to notice that the limit as  $x \rightarrow 0$  of the functions  $G_3^{\text{scalar}}, G_6^{\text{scalar}}, G_7^{\text{scalar}}, G_8^{\text{scalar}}$  and  $G_9^{\text{scalar}}$  vanish identically, that is,

$$G_3^{\text{scalar}}(0, y) = G_6^{\text{scalar}}(0, y) = G_7^{\text{scalar}}(0, y) = G_8^{\text{scalar}}(0, y) = G_9^{\text{scalar}}(0, y) = 0. \quad (5.78)$$

The explicit expression for the remaining non-vanishing  $G_i^{\text{scalar}}$  for pure electric field in  $n$  dimensions is

$$G_0^{\text{scalar}}(0, y) = -\pi^{-n/2} \text{Li}_{\frac{n}{2}}(-e^{-\pi y}), \quad (5.79)$$

$$G_1^{\text{scalar}}(0, y) = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi^{n/2-1}} \text{Li}_{\frac{n}{2}-1}(-e^{-\pi y}), \quad (5.80)$$

$$\begin{aligned} G_2^{\text{scalar}}(0, y) = & -\frac{1}{48\pi^{n/2+1}} \left\{ 2\pi^3 y \text{Li}_{\frac{n}{2}-2}(-e^{-\pi y}) + (n+4)\pi^2 \text{Li}_{\frac{n}{2}-1}(-e^{-\pi y}) \right. \\ & \left. - 18\pi y \text{Li}_{\frac{n}{2}}(-e^{-\pi y}) - 9(n+2) \text{Li}_{\frac{n}{2}+1}(-e^{-\pi y}) \right\}, \end{aligned} \quad (5.81)$$

$$G_4^{\text{scalar}}(0, y) = \frac{1}{384\pi^{n/2+1}} \left\{ -16\pi^3 y \text{Li}_{\frac{n}{2}-2}(-e^{-\pi y}) - 4\pi^2(2n+9-3y^2) \text{Li}_{\frac{n}{2}-1}(-e^{-\pi y}) \right. \\ \left. + 12(n+12)\pi y \text{Li}_{\frac{n}{2}}(-e^{-\pi y}) + 3(n+2)(n+24) \text{Li}_{\frac{n}{2}+1}(-e^{-\pi y}) \right\}, \quad (5.82)$$

$$G_5^{\text{scalar}}(0, y) = \frac{1}{256\pi^{n/2+1}} \left\{ 4\pi^2(1-3y^2) \text{Li}_{\frac{n}{2}-1}(-e^{-\pi y}) - 12n\pi y \text{Li}_{\frac{n}{2}}(-e^{-\pi y}) \right. \\ \left. - 3n(n+2) \text{Li}_{\frac{n}{2}+1}(-e^{-\pi y}) \right\}. \quad (5.83)$$

In the physical case of  $n = 4$  some of the polylogarithmic functions can be expressed in terms of elementary functions. In this case we have

$$G_0^{\text{scalar}}(0, y) = -\frac{1}{\pi^2} \text{Li}_2(-e^{-\pi y}). \quad (5.84)$$

$$G_1^{\text{scalar}}(0, y) = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi} \ln(1 + e^{-\pi y}), \quad (5.85)$$

$$G_2^{\text{scalar}}(0, y) = \frac{1}{48\pi^3} \left\{ \frac{2\pi^3 y e^{-\pi y}}{1 + e^{-\pi y}} + 8\pi^2 \ln(1 + e^{-\pi y}) \right. \\ \left. + 18\pi y \text{Li}_2(-e^{-\pi y}) + 54 \text{Li}_3(-e^{-\pi y}) \right\}, \quad (5.86)$$

$$G_4^{\text{scalar}}(0, y) = \frac{1}{384\pi^3} \left\{ \frac{16\pi^3 y e^{-\pi y}}{1 + e^{-\pi y}} + 4\pi^2(17 - 3y^2) \ln(1 + e^{-\pi y}) \right. \\ \left. + 192\pi y \text{Li}_2(-e^{-\pi y}) + 504 \text{Li}_3(-e^{-\pi y}) \right\}, \quad (5.87)$$

$$G_5^{\text{scalar}}(0, y) = -\frac{1}{256\pi^3} \left\{ 4\pi^2(1-3y^2) \ln(1 + e^{-\pi y}) + 48\pi y \text{Li}_2(-e^{-\pi y}) \right. \\ \left. + 72 \text{Li}_3(-e^{-\pi y}) \right\}. \quad (5.88)$$

We study now the behavior of these functions as  $y \rightarrow 0$ , which corresponds to the limit

$$B = 0, \quad m^2 \ll E. \quad (5.89)$$

By taking the limit as  $y \rightarrow 0$  of the expression (5.77) and by noticing that

$$\text{Li}_n(-1) = -(1 - 2^{1-n})\zeta(n), \quad (5.90)$$

where  $\zeta(x)$  denotes the Riemann zeta function, we obtain

$$G_0^{\text{scalar}}(0, 0) = \frac{(1 - 2^{1-n/2})}{\pi^{n/2}} \zeta\left(\frac{n}{2}\right). \quad (5.91)$$

Next, by taking the limit as  $y \rightarrow 0$  and by using the formula (5.90), it is not difficult to obtain

$$G_1^{\text{scalar}}(0, 0) = -\left(\frac{1}{6} - \xi\right) \pi^{1-n/2} (1 - 2^{2-n/2}) \zeta\left(\frac{n}{2} - 1\right), \quad (5.92)$$

$$\begin{aligned} G_2^{\text{scalar}}(0, 0) = & -\frac{1}{48\pi^{n/2+1}} \left\{ -(n+4)\pi^2 (1 - 2^{2-n/2}) \zeta\left(\frac{n}{2} - 1\right) \right. \\ & \left. + 9(n+2)(1 - 2^{-n/2}) \zeta\left(\frac{n}{2} + 1\right) \right\}, \end{aligned} \quad (5.93)$$

$$\begin{aligned} G_4^{\text{scalar}}(0, 0) = & \frac{1}{384\pi^{n/2+1}} \left\{ 4\pi^2 (2n+9)(1 - 2^{2-n/2}) \zeta\left(\frac{n}{2} - 1\right) \right. \\ & \left. - 3(n+2)(n+24)(1 - 2^{-n/2}) \zeta\left(\frac{n}{2} + 1\right) \right\}, \end{aligned} \quad (5.94)$$

$$\begin{aligned} G_5^{\text{scalar}}(0, 0) = & \frac{1}{256\pi^{n/2+1}} \left\{ -4\pi^2 (1 - 2^{2-n/2}) \zeta\left(\frac{n}{2} - 1\right) \right. \\ & \left. + 3n(n+2)(1 - 2^{-n/2}) \zeta\left(\frac{n}{2} + 1\right) \right\}. \end{aligned} \quad (5.95)$$

We consider, at this point, the physical case of four dimensions. By setting  $n = 4$  in (5.91) we obtain

$$G_0^{\text{scalar}}(0, 0) = \frac{1}{12}. \quad (5.96)$$

Now, we notice the following relation

$$(1 - 2^{2-n/2}) \zeta\left(\frac{n}{2} - 1\right) = \eta\left(\frac{n}{2} - 1\right), \quad (5.97)$$

where  $\eta(x)$  is the Dirichlet eta function. In the particular case of four dimensions we have that

$$\lim_{n \rightarrow 4} (1 - 2^{2-n/2}) \zeta\left(\frac{n}{2} - 1\right) = \eta(1) = \ln 2. \quad (5.98)$$

By using the last remark we obtain the values of the functions  $G_i(n, y)$  in four dimensions

$$G_1^{\text{scalar}}(0, 0) = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi} \ln 2, \quad (5.99)$$

$$G_2^{\text{scalar}}(0, 0) = \frac{1}{6\pi} \ln 2 - \frac{27}{32\pi^3} \zeta(3), \quad (5.100)$$

$$G_4^{\text{scalar}}(0, 0) = \frac{17}{96\pi} \ln 2 - \frac{63}{64\pi^3} \zeta(3), \quad (5.101)$$

$$G_5^{\text{scalar}}(0, 0) = -\frac{1}{64\pi} \ln 2 + \frac{27}{128\pi^3} \zeta(3). \quad (5.102)$$

### 5.3.2 Spinor Fields

For spinor fields the expressions for the non-vanishing  $G_i^{\text{spinor}}$  in the limit  $x \rightarrow 0$  are

$$G_0^{\text{spinor}}(0, y) = 2^{[n/2]} \pi^{-n/2} \text{Li}_{n/2}(e^{-\pi y}) \quad (5.103)$$

$$G_1^{\text{spinor}}(0, y) = -\frac{2^{[n/2]}}{12} \frac{1}{\pi^{n/2-1}} \text{Li}_{\frac{n}{2}-1}(e^{-\pi y}), \quad (5.104)$$

$$\begin{aligned} G_2^{\text{spinor}}(0, y) = & -\frac{2^{[n/2]}}{48\pi^{n/2+1}} \left\{ 2\pi^3 y \text{Li}_{\frac{n}{2}-2}(e^{-\pi y}) + (n+4)\pi^2 \text{Li}_{\frac{n}{2}-1}(e^{-\pi y}) \right. \\ & \left. - 18\pi y \text{Li}_{\frac{n}{2}}(e^{-\pi y}) - 9(n+2) \text{Li}_{\frac{n}{2}+1}(e^{-\pi y}) \right\}, \end{aligned} \quad (5.105)$$

$$\begin{aligned} G_4^{\text{spinor}}(0, y) = & \frac{2^{[n/2]}}{384\pi^{n/2+1}} \left\{ -16\pi^3 y \text{Li}_{\frac{n}{2}-2}(e^{-\pi y}) - 4\pi^2 (2n+12-3y^2) \text{Li}_{\frac{n}{2}-1}(e^{-\pi y}) \right. \\ & \left. + 12(n+12)\pi y \text{Li}_{\frac{n}{2}}(e^{-\pi y}) + 3(n+2)(n+24) \text{Li}_{\frac{n}{2}+1}(e^{-\pi y}) \right\}, \end{aligned} \quad (5.106)$$

$$\begin{aligned} G_5^{\text{spinor}}(0, y) = & -2^{[n/2]} \frac{3}{256\pi^{n/2+1}} \left\{ 4\pi^2 (4+y^2) \text{Li}_{\frac{n}{2}-1}(e^{-\pi y}) + 4n\pi y \text{Li}_{\frac{n}{2}}(e^{-\pi y}) \right. \\ & \left. + n(n+2) \text{Li}_{\frac{n}{2}+1}(e^{-\pi y}) \right\}. \end{aligned} \quad (5.107)$$

In the particular case of  $n = 4$  the above results read

$$G_0^{\text{spinor}}(0, y) = \frac{4}{\pi^2} \text{Li}_2(e^{-\pi y}) , \quad (5.108)$$

$$G_1^{\text{spinor}}(0, y) = \frac{1}{3\pi} \ln(1 - e^{-\pi y}) , \quad (5.109)$$

$$\begin{aligned} G_2^{\text{spinor}}(0, y) = & -\frac{1}{12\pi^3} \left\{ \frac{2\pi^3 y e^{-\pi y}}{1 - e^{-\pi y}} - 8\pi^2 \ln(1 - e^{-\pi y}) \right. \\ & \left. - 18\pi y \text{Li}_2(e^{-\pi y}) - 54\text{Li}_3(e^{-\pi y}) \right\} , \end{aligned} \quad (5.110)$$

$$\begin{aligned} G_4^{\text{spinor}}(0, y) = & -\frac{1}{96\pi^3} \left\{ \frac{16\pi^3 y e^{-\pi y}}{1 - e^{-\pi y}} - 4\pi^2 (20 - 3y^2) \ln(1 - e^{-\pi y}) \right. \\ & \left. - 192\pi y \text{Li}_2(e^{-\pi y}) - 504\text{Li}_3(e^{-\pi y}) \right\} , \end{aligned} \quad (5.111)$$

$$G_5^{\text{spinor}}(0, y) = \frac{3}{16\pi^3} \left\{ \pi^2 (4 + y^2) \ln(1 - e^{-\pi y}) - 4\pi y \text{Li}_2(e^{-\pi y}) - 6\text{Li}_3(e^{-\pi y}) \right\} . \quad (5.112)$$

In the case of spinor fields, for  $n > 4$ , there is a well defined limit as  $y \rightarrow 0$ . In fact, by taking the massless limit,  $y \rightarrow 0$ , of the expression (5.103) and noticing that

$$\text{Li}_n(1) = \zeta(n) , \quad (5.113)$$

we obtain

$$G_0^{\text{spinor}}(0, 0) = \frac{2^{[n/2]}}{\pi^{n/2}} \zeta\left(\frac{n}{2}\right) . \quad (5.114)$$

Analogously, in the limit as  $y \rightarrow 0$  the result for the remaining  $G_i^{\text{spinor}}$  can be written as follows

$$G_1^{\text{spinor}}(0, 0) = -\frac{2^{[n/2]}}{12} \pi^{1-n/2} \zeta\left(\frac{n}{2} - 1\right) , \quad (5.115)$$

$$\begin{aligned} G_2^{\text{spinor}}(0, 0) = & -\frac{2^{[n/2]}}{48\pi^{n/2+1}} \left\{ (n+4)\pi^2 \zeta\left(\frac{n}{2} - 1\right) - 9(n+2)\zeta\left(\frac{n}{2} + 1\right) \right\} , \\ & (5.116) \end{aligned}$$

$$G_4^{\text{spinor}}(0,0) = \frac{2^{[n/2]}}{384\pi^{n/2+1}} \left\{ -4\pi^2(2n+12)\zeta\left(\frac{n}{2}-1\right) + 3(n+2)(n+24)\zeta\left(\frac{n}{2}+1\right) \right\}, \quad (5.117)$$

$$G_5^{\text{spinor}}(0,0) = -2^{[n/2]} \frac{3}{256\pi^{n/2+1}} \left\{ 16\pi^2\zeta\left(\frac{n}{2}-1\right) + n(n+2)\zeta\left(\frac{n}{2}+1\right) \right\}. \quad (5.118)$$

We turn our attention, now, to the physical case of  $n = 4$ . From the expression in (5.114) we obtain the following result

$$G_0^{\text{spinor}}(0,0) = \frac{2}{3}. \quad (5.119)$$

It is evident, from the expressions in (5.109)-(5.112), that the functions  $G_i^{\text{spinor}}(0,y)$  in four dimensions represent a special case since there is an infrared divergence as  $m \rightarrow 0$  (or  $y \rightarrow 0$ ). This means that there is no well-defined value for the massless limit  $y \rightarrow 0$ . Instead, we find a logarithmic divergence,  $\log(\pi y)$ . In order to analyze this case we set  $n = 4$  from the beginning in the expressions for finite  $y$ , and then we examine the asymptotics as  $y \rightarrow 0$ . By using the equations (5.109)-(5.112) we obtain

$$G_1^{\text{spinor}}(0,y) = \frac{1}{3\pi} \log(\pi y) + O(y), \quad (5.120)$$

$$G_2^{\text{spinor}}(0,y) = \frac{2}{3\pi} \log(\pi y) - \frac{1}{6\pi} + \frac{9}{2\pi^3} \zeta(3) + O(y), \quad (5.121)$$

$$G_4^{\text{spinor}}(0,y) = \frac{5}{6\pi} \log(\pi y) - \frac{1}{6\pi} + \frac{21}{4\pi^3} \zeta(3) + O(y), \quad (5.122)$$

$$G_5^{\text{spinor}}(0,y) = \frac{3}{4\pi} \log(\pi y) + \frac{9}{8\pi^3} \zeta(3) + O(y). \quad (5.123)$$

Notice that, in four dimensions the functions  $G_i^{\text{spinor}}(x,y)$  are singular at the point  $x = y = 0$ . In particular, the limits  $x \rightarrow 0$  and  $y \rightarrow 0$  are not commutative, that is, the limits as  $x \rightarrow 0$  of the eqs. (5.64)-(5.73) (obtained as  $y \rightarrow 0$  for a finite  $x$ ) are different from the eqs. (5.120)-(5.123) (obtained as  $y \rightarrow 0$  for  $x = 0$ ).

## 6 Concluding Remarks

In this paper we have continued the study of the heat kernel and the effective action for complex (scalar and spinor) quantum fields in a strong constant electromagnetic field and a gravitational field initiated in [14]. We study here an *essentially non-perturbative regime* when the electromagnetic field is so strong that one has to take into account all its orders. In this situation the standard asymptotic expansion of the heat kernel does not apply since the electromagnetic field can not be treated as a perturbation. In [14] we established the existence of a new non-perturbative asymptotic expansion of the heat kernel and computed explicitly the first three coefficients of this expansion.

In the present paper we computed the first two coefficients (of zero and the first order in the Riemann curvature) explicitly in  $n$ -dimensions by using the spectral decomposition of the electromagnetic field tensor. We applied this result for the calculation of the effective action in the physical pseudo-Euclidean (Lorentzian) case and computed explicitly the imaginary part of the effective action both in the general case and in the cases of physical interest. We also computed the asymptotics of the obtained results for supercritical electric fields.

We have discovered a *new infrared divergence* in the imaginary part of the effective action for massless spinor fields in four dimensions (or supercritical electric field), which is induced purely by the gravitational corrections. This means physically that the creation of massless spinor particles (or massive particles in supercritical electric field) is magnified substantially by the presence of the gravitational field. Further analysis shows that a similar effect occurs for any massless fields (also scalar fields) in the second order in the Riemann curvature. This effect could have important consequences for theories with spontaneous symmetry breakdown when the mass of charged particles is generated by a Higgs field. Such theories would exhibit a significant amount of created particles (in the massless limit an infinite amount) at the phase transition point when the symmetry is restored and the massive charged particles become massless. That is why this seems to be an interesting *new physical effect* that deserves further investigation.

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